Stochastic Control Theory Exercise 1

Due: December 16, 2019

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with an increasing filtration $(\mathcal{F}_t)_{t\geq 0}$.

Part 1: Definition and Properties of Stochastic Integrals

Let $(W(t))_{t\geq 0}$ be an $\mathcal{F}_t$-adapted real-valued Wiener process. We denote by $L^2(T)$ the set of real-valued stochastic processes $(X(t))_{t\in[0,T]}$ such that

- the mapping $(t,\omega) \to X(t,\omega)$ is $\mathcal{B}\times\mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,T]$;
- $(X(t))_{t\in[0,T]}$ is $\mathcal{F}_t$-adapted;
- $\mathbb{E}\int_0^T |X(t)|^2 dt < \infty$.

Then the Itô stochastic integral

$$\int_0^T X(t)\, dW(t)$$

is well defined for all $X \in L^2(T)$.

Exercises:

1. Show that for all $a, b \in \mathbb{R}$ and all $X, Y \in L^2(T)$

$$\int_0^T a\, X(t) + b\, Y(t)\, dW(t) = a \int_0^T X(t)\, dW(t) + b \int_0^T Y(t)\, dW(t).$$

2. Verify that for all $X \in L^2(T)$

$$\mathbb{E}\left[\int_0^T X(t)\, dW(t)\right] = 0.$$

3. Prove that the Itô isometry

$$\mathbb{E}\left[\left(\int_0^T X(t)\, dW(t)\right)^2\right] = \mathbb{E}\left[\int_0^T X^2(t) dt\right]$$

   holds for all $X \in L^2(T)$.

4. For all $r, t \in [0,T]$ with $r \leq t$ and all $X \in L^2(T)$, we define

$$\int_r^t X(s)\, dW(s) = \int_0^T \mathbb{1}_{[r,t]}(s)\, X(s)\, dW(s),$$

   where $\mathbb{1}$ denotes the indicator function. Derive that

$$\int_0^t X(s)\, dW(s) = \int_0^r X(s)\, dW(s) + \int_r^t X(s)\, dW(s).$$

5. Show that the process $(\int_0^t X(s)\, dW(s))_{t\in[0,T]}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Hint: Verify these properties for elementary processes in $L^2(T)$ and use a suitable density result.

Part 2: The Itô Formula and Stochastic Differential Equations

We first introduce the Itô formula.

Theorem. *Let the real-valued stochastic process* $(X(t))_{t\in[0,T]}$ *be of the form*

$$X(t) = x_0 + \int_0^t A(s)\,ds + \int_0^t B(s)\,dW(s),$$

where x_0 *is a* $\mathcal{F}_0$*-measurable real-valued random variable and the stochastic processes* $(A(t))_{t\in[0,T]}$ *and* $(B(t))_{t\in[0,T]}$ *are* $\mathcal{F}_t$*-adapted such that*

$$\mathbb{E}\left[\int_0^T |A(t)|\,dt\right] + \mathbb{E}\left[\int_0^T |B(t)|^2\,dt\right] < \infty.$$

Assume that $f \in C^{1,2}([0,T)\times\mathbb{R})$*, i.e. the function* f *is once continuous differentiable with with respect to the first argument and twice continuous differentiable with respect to the second argument. Then we have*

$$f(t,X(t)) = f(0,x_0) + \int_0^t \left[\frac{\partial}{\partial t} f(s,X(s)) + A(s)\frac{\partial}{\partial x} f(s,X(s)) + \frac{1}{2}B^2(s)\frac{\partial^2}{\partial x^2} f(s,X(s))\right] ds$$
$$+ \int_0^t B(s)\frac{\partial}{\partial x} f(s,X(s))\,dW(s).$$

For a real-valued stochastic process $(X(t))_{t\in[0,T]}$, we introduce the stochastic differential equation (SDE)

$$\begin{cases} dX(t) = b(t,X(t))\,dt + \sigma(t,X(t))\,dW(t) \\ X(0) = x_0, \end{cases}$$

where x_0 is a $\mathcal{F}_0$-measurable real-valued random variable and $(W(t))_{t\geq 0}$ is a $\mathcal{F}_t$-adapted real-valued Wiener process. The mappings $b,\sigma\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ satisfy some suitable properties (growth condition and Lipschitz condition) such that the SDE has a unique solution given by

$$X(t) = \int_0^t b(s,X(s))\,ds + \int_0^t \sigma(s,X(s))\,dW(s).$$

Exercises:

1. Let $\mu,\sigma \in L^\infty([0,T])$. Verify that the solution of the SDE
$$\begin{cases} dX(t) = \mu(t)X(t)\,dt + \sigma(t)X(t)\,dW(t) \\ X(0) = x_0, \end{cases}$$
is given by
$$X(t) = x_0 \exp\left\{\int_0^t \mu(s) - \frac{1}{2}\sigma^2(s)\,ds + \int_0^t \sigma(s)\,dW(s)\right\}.$$

2. Let the processes $(X_1(t))_{t\in[0,T]}$ and $(X_2(t))_{t\in[0,T]}$ denote the coordinates of a Wiener process $(W(t))_{t\geq 0}$ on the unit circle defined by
$$X_1(t) = \cos(W(t)), \quad X_2(t) = \sin(W(t)).$$
Find SDEs for $(X_1(t))_{t\in[0,T]}$ and $(X_2(t))_{t\in[0,T]}$.

3. Let $(X(t))_{t\in[0,T]}$ be the mean-reverting Ornstein-Uhlenbeck process given by

$$\begin{cases} dX(t) = [m - X(t)]\,dt + \sigma\,dW(t) \\ X(0) = x_0, \end{cases}$$

where $m, \sigma, x_0 \in \mathbb{R}$ are constants. Find the solution of this SDE and calculate the mean $\mathbb{E}[X(t)]$ and the variance $\mathrm{Var}[X(t)]$ for all $t \in [0,T]$.

4. Find a solution $(X(t))_{t\in[0,T]}$ of the nonlinear SDE

$$\begin{cases} dX(t) = X^{\gamma}(t)\,dt + \sigma X(t)\,dW(t) \\ X(0) = x_0, \end{cases}$$

where $\gamma, \sigma \in \mathbb{R}$ are constants.
Hint: It is allowed to use the product rule for Itô process as well as solutions for determinitstic ODEs.

Part 3: Numerics for Stochastic Control Problems

Let the real-valued stochastic process $(X(t))_{t\in[0,T]}$ be the solution of the controlled SDE

$$\begin{cases} dX(t) = b(t, X(t), u(t))\,dt + \sigma(t, X(t), u(t))\,dW(t) \\ X(0) = x_0, \end{cases}$$

where x_0 is a $\mathcal{F}_0$-measurable real-valued random variable and $(W(t))_{t\geq 0}$ is a $\mathcal{F}_t$-adapted real-valued Wiener process. The real-valued stochastic process $(u(t))_{t\in[0,T]}$ denotes the control satisfying $\mathbb{E}\int_0^T |u(t)|^2 dt < \infty$. We introduce a partition of the time interval $[0,T]$ such that

$$0 = t_0 < t_1 < ... < t_N = T$$

and $t_{k+1} - t_k = \Delta t > 0$ for all $k = 0, 1, ..., N-1$. Then the Euler–Maruyama method provides a numerical solution for SDEs. One introduces the iteration scheme

$$\hat{X}_{k+1} = \hat{X}_k + b(t_k, \hat{X}_k, u(t_k))\Delta t + \sigma(t_k, \hat{X}_k, u(t_k))\Delta W_k$$

for each $k = 0, 1, ..., N-1$ with $\hat{X}_0 = x_0$ and $\Delta W_k = W(t_{k+1}) - W(t_k) \sim \mathcal{N}(0, \Delta t)$.

PC - Exercises:

1. Simulate the Wiener process $(W(t))_{t\geq 0}$ on the time interval $[0,T]$ with $T = 1$. Plot t against $W(t)$.

2. Implement the Euler–Maruyama method on the time interval $[0,T]$ for the SDE

$$\begin{cases} dX(t) = \mu X(t)\,dt + \sigma X(t)\,dW(t) \\ X(0) = x_0, \end{cases}$$

where $\mu, \sigma \in \mathbb{R}$. Use the values $T = 1$, $x_0 = 1$, $\mu = 2$, and $\sigma = 1$. For the step size use $\Delta t = 2^{-2}$, 2^{-4}, 2^{-6}, and 2^{-8}. From Exercise 1 in Part 2, we know that the explicit solution of the SDE is given by

$$X(t) = x_0 \exp\left\{\mu t - \frac{1}{2}\sigma^2 t + \sigma\,W(t)\right\}.$$

Plot the explicit solution and the numerical solution using the same trajectory of the Wiener process.

3. In Section 1.1 in the lecture, we considered the wealth process $(X(t))_{t\in[0,T]}$ satisfying

$$dX(t) = ([r + (\mu - r)\pi(t)]X(t) - c(t))\,dt + \sigma\,\pi(t)X(t)\,dW(t).$$

Implement the Euler–Maruyama method on the time interval $[0, T]$. Use the values $T = 1$, $x_0 = 20$, $r = 1$, $\mu = 2$, and $\sigma = 1$. For the trading strategy $\pi(t)$ use the constants values $0, 0.1, 0.5, 0.7$, and 1. For the consumption plan $c(t)$ use the constants values $0, 0.5, 1, 1.5$, and 2. Calculate the values of the cost functional

$$J(X, \pi, c) = \mathbb{E}\left[\int_0^T e^{-\delta t}\frac{1}{\gamma}c^{\gamma}(t)\,dt + e^{-\delta T}\frac{1}{\gamma}X^{\gamma}(T)\right]$$

with $\delta = \gamma = 0.5$.

Literature:

- P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer, 1992
- P. E. Protter, *Stochastic Integration and Differential Equations*, Springer, 2005
- K. Sobczyk, *Stochastic Differential Equations, With Applications to Physics and Engineering*, Springer, 1991
- B. Øksendal, *Stochastic Differential Equations, An Introduction with Applications*, Springer, 2013